Recent advances in the study of non local models in population dynamics

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Outline of this talk

1· Heterogeneous dispersal process

1. Context/Motivation
2. Results
3. Idea of the Proofs

2· Study of the impact of some agricultural management on non target species

1. Context/Motivation
2. Results
3. Idea of the Proofs

I. Heterogeneous dispersal

Main motivation

Analysis of gene flux in a population of tree (E.Klein)

- Complex Landscape
- Want to understand the dynamics of a tree species ?
- Understand the genetic structure after colonisation ?

Main feature

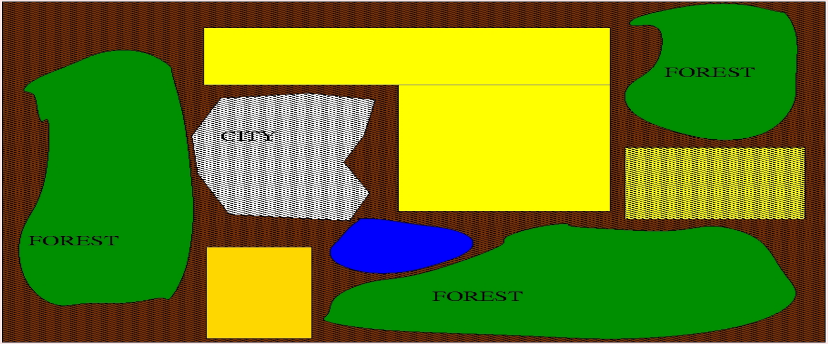
Using genetics markers you have the technology to acquire data and analyse it statistically in order to say which is the mother/father of whom and describe the dynamics in terms of a dispersal kernel.

Questions asked

Modelling of simple heterogeneous Dispersal which is usable to do Bayesian Statistics ?

Continuous Model in population dynamic

Aim's :



Describe accurately the evolution of a species in a natural environment with a good feed back from the collected data

Main assumptions :

- The dispersal of an individual is governed by a probability density intrinsic to the species.
- The environment is heterogeneous and affect the dispersal in all possible ways.

The Model considered :

The dispersal process considered

$$\frac{\partial u}{\partial t} = \int_\Omega K\left(\frac{x_1 - y_1}{h_1(x)g_1(y)}; \ldots ; \frac{x_n - y_n}{h_n(x)g_n(y)}\right) u(y)\,dy - c(x)u$$

$$u(0, x) = u_0$$

$\Omega \subset \mathbb{R}^n$is an open subset.

- $c(x) := \int_\Omega K\left(\frac{y_1 - x_1}{g_1(x)h_1(y)}; \ldots ; \frac{y_n - x_n}{g_n(x)h_n(y)}\right) dy.$
- $K \in C_c(\mathbb{R}^n)$, $K \geq 0$, $\|K\|_1 = 1$, $supp(K) := B(0, 1)$
- $h_i, g_i \in C(\Omega)$, $h_i, g_i \geq 0$ describe the impact of the environment on the dispersal process

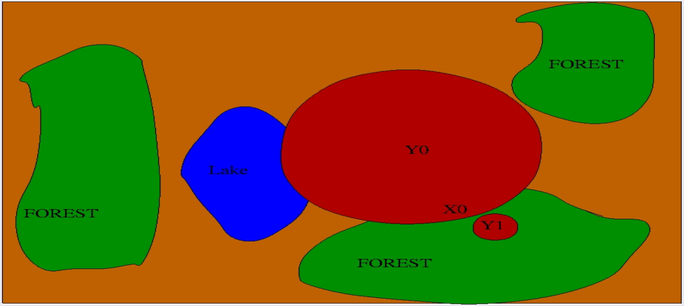
Interpretation of the impact of g and h

- g represents the constrain of the environment to the displacement.
- h can be assimilated to a measure of attractiveness of a site.

Effects of the environment on the dispersal

To simplify let $g_i = g_j$. Fix $x_0 \in \Omega$, then to jump to the position x_0, the individuals at the position $y \in \Omega$ must verify

$$\left\| \frac{x_0 - y}{g(y)} \right\| \leq 1 \iff y \in B(x_0, g(y)) \iff x_0 \in B(y, g(y)).$$



$g(y) = 0$: No migration for those individuals

$g(y) \approx 0$: local migration

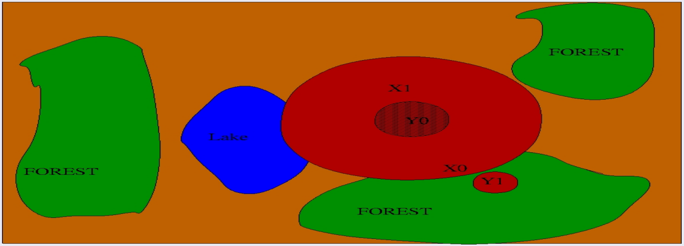
$g(y) >> 1$: Long range migration

$g(y) = +\infty$: No constraint of distance

Effects of the environment on the dispersal

To simplify let $g_i = g_j$ and $h_i = h_j$. Fix $x_0 \in \Omega$, then to jump to the position x_0, the individuals at the position $y \in \Omega$ must verify

$$\left\| \frac{x_0 - y}{h(x_0)g(y)} \right\| \leq 1 \iff y \in B(x_0, h(x_0)g(y)) \iff x_0 \in B(y, h(x_0)g(y)).$$



- $h(x)g(y) = 0$: No migration for those individuals to the location x
- $h(x)g(y) \approx 0$: local migration to the location x
- $h(x)g(y) >> 1$: Long range migration to the location x
- $h(x)g(y) = +\infty$: No constraint of distance

Results

Known Results :

- Ω bounded, $g, h > 0$ standard theory
- Homogeneous dispersal processes ($g_i = g_j \equiv Cste$) (Rossi, Chasseigne, Chavez)
- General kernel with a detailed balanced condition (Perthame, Mischler, ...)
- $h \equiv Cste$, (Cortazar, C., Elgueta, Martinez, Garcia-Mellian,...)

Results (C., Martinez 2012) :

- **Characterisation of the asymptotic behaviour of the solution u in terms of the behaviour of h, g.**

Let $g > 0$, $h \geq 0$ bounded. Then, $\forall\ u_0 \in L^1 \cap L^\infty$,

- **The bounded Case**
  (i) then
  $$u(x,t) \to p(x)$$
  where $p(x)$ is the positive principal eigenfunction
- **The unbounded Case**
  (ii) For $h \equiv Cste$ in $\mathbb{R}^n \setminus \Omega'$ where Ω' is bounded then
  $$\forall\ x\ \in \mathbb{R}\ \lim_{t\to\infty} u(x,t) = 0.$$

Idea of the proof

Lemma 1

If $u_0 \in L^1 \cap L^\infty$ *then*

$$\|u_0\|_1 = \|u\|_1, \quad \|u\|_\infty \leq C.$$

Lemma 2

There exists a positive stationary solution p of the problem i.e. p solves :

$$\int_\Omega K(x,y)p(y)\,dy - c(x)p = 0$$

Moreover :

- *If* Ω *is bounded, p is unique (up to a multiplicative constant) and bounded*
- *If* $\Omega = \mathbb{R}$ *and* $h \equiv Cste$ *in* $\mathbb{R} \setminus \Omega'$ *then* $\inf_{\mathbb{R}} p > c_0 \geq 0$

Lemma 3

The quantity $\frac{u^2}{p}$ *satisfies :*

$$\frac{\partial}{\partial t}\int_\Omega \frac{u^2}{p} = -\int_\Omega\int_\Omega K(x,y)p(y)\left[\frac{u(y)}{p(y)} - \frac{u(x)}{p(x)}\right]^2 dxdy.$$

Idea of the proof of Lemma 2

sketch of the proof when the $g_i \geq \alpha_i > 0$, $h_i > 0$

- Ω bounded
  - $\mathcal{M}[\phi] := \frac{1}{k(x)} \int_\Omega K(x,y)\phi(y)\,dy$ is a positive compact operator
  - Krein-Rutman + Mass preservation $\Longrightarrow \phi_p$ is our desired solution
  - Convergence using the classical Relative Entropy method

- Ω unbounded
  - $(\Omega_n)_{n\in\mathbb{N}}$ sequence of bounded domain, so that $\Omega_n \to \Omega$
  - $(\phi_n)_{n\in\mathbb{N}}$ the associated stationary solution to a truncated problem set on Ω_n
  - Harnack type inequality (C 2011), i.e, For all compact set $\omega \subset\subset \Omega$, $\exists C1 > 0$, so that for all stationary positive bounded solution v

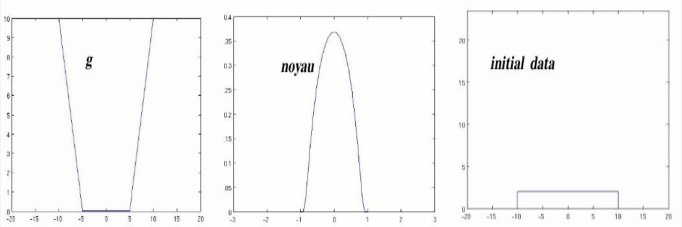
$$\sup_\omega v(x) \leq C \inf_\omega v(x).$$

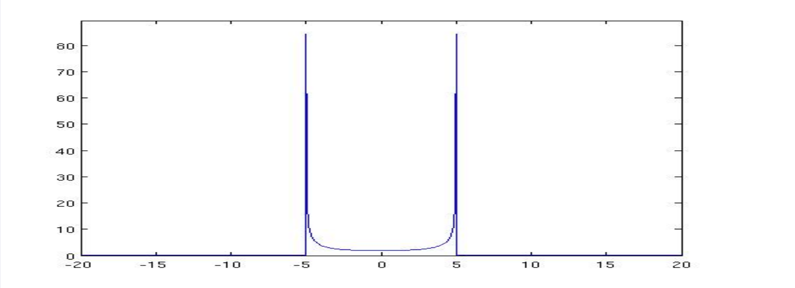
  - For $\Omega = \mathbb{R}$, another Harnack type inequality (CCEM, 2007), i.e. $\exists C, D > 0$ so that for all stationary positive bounded solution v then $\forall x, y \in \mathbb{R}$,

$$v(x) \leq C \int_{y-D}^{y+D} v(s)\,ds.$$

Example of a numerical simulation

"Modelling of a hole" ($g \approx 0$), $h \equiv 1$

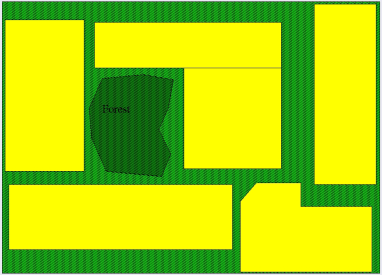


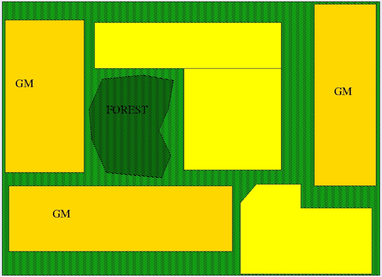


II. Study of the impact of an agricultural practice on non target species (ERC Project AMIGA, A. Mesean)

Context of the study

Schematics of a typical agricultural landscape





The GM fields are target some particular pest.

Questions

- How the introduction of the GM fields affects the pest ?
- What are the side effects on **non target species** (species which belongs to the same group of the pest but is harmless) ?
- What are the consequence of GM fields on biodiversity ?
- Is there any way to control the proportion of GM fields to minimize the damage and conserve biodiversity and maximize some agricultural benefit ?

The refuge model considered

$$\frac{\partial u}{\partial t} = \mathcal{K}_\Omega[u] + a_0(x)u + \lambda a_1(x)u - \beta(x)u^p \qquad (1)$$

where Ω is a bounded set of $\mathbb{R}^n$, $\lambda \in \mathbb{R}, p > 1$ and

- $a_0, a_1, \beta \in C(\bar{\Omega})$, $a_i, \beta \geq 0$
- $\mathcal{K}$ is an integral operator describing the dispersal of the density of individuals

$$\mathcal{K}[u] := \int_\Omega K(x,y)u(y)\,dy - u(x)\int_\Omega K(y,x)\,dy$$

with $K \in C(\bar{\Omega}, \bar{\Omega})$, $K \geq 0$ so that

$$\exists c_0 > 0, \epsilon_0 > 0 \quad \text{such that} \inf_{x\in\Omega}\left(\inf_{y \in B(x,\epsilon_0)} K(x,y)\right) > c_0.$$

Typical example : $K(x,y) = J\left(\frac{x_1 - y_1}{h_1(x)g_1(y)}; \ldots; \frac{x_n - y_n}{h_n(x)g_n(y)}\right)$.

Interpretation

- $K(x,y)$ describes the probability to jump from a site y to a site x. We assume that
- a_0, β represent an intrinsic growth and death rate
- λa_1 represent controlled area ($\sim$The GM field)

Known Results

- For operator $\mathcal{K}$ is replace by a diffusion operator $\mathcal{L} := a_{ij}(x)\partial_{ij} + b_i\partial_i + c(x)$ (Ouang, Fraile, Garcia-mellan, Lopez-Gomez, Koch- Medina,Du, Ma...).
- For homogeneous symmetric nonlocal $\mathcal{K}$(i.e. $\mathcal{K}_\Omega[u] := \int_\Omega J(x-y)u(y)\,dy - u$) (Garcia-mellan, Rossi)

Absence of refuge : i.e $\beta > 0$

There exists λ^* so that there exists a unique positive stationary solution to (1) **if and only if** $\lambda > \lambda^*$. Moreover, the map $\lambda \to u_\lambda$ is monotone increasing and we have

$$\forall x \in \bar{\Omega} \quad \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty$$

$$\forall x \in \bar{\Omega} \quad \lim_{\lambda \to \lambda^{*,+}} u_\lambda = 0$$

Existence of a refuge : i.e $\beta_{|\omega} \equiv 0$.

There exists $\lambda^* < \lambda^{**}$ that there exists a bounded solution to (1) **if and only if** $\lambda^{**} > \lambda > \lambda^*$. Moreover, the map $\lambda \to u_\lambda$ is monotone increasing and we have

$$\forall x \in \bar{\Omega} \quad \lim_{\lambda \to \lambda^{**,-}} u_\lambda(x) = +\infty$$

$$\forall x \in \bar{\Omega} \quad \lim_{\lambda \to \lambda^{*,+}} u_\lambda(x) = 0$$

results

Absence of refuge : i.e $\beta > 0$ C, 2012

For K, a_i, β as above then there exists $\lambda^* \in [-\infty; \infty)$, so that for all $\lambda > \lambda^*$ there exists a unique positive stationary solution u_λ of (1) and when $\lambda^* \in \mathbb{R}$, there is no positive stationary solution to (1) for all $\lambda \leq \lambda^*$. Moreover, the map $\lambda \to u_\lambda$ is monotone increasing and we have

$$\forall x \in \bar{\Omega} \quad \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty$$

$$\forall x \in \bar{\Omega} \quad \lim_{\lambda \to \lambda^{*,+}} u_\lambda(x) = u_\infty$$

where u_∞ is the unique non negative solution of

$$\int_{\Omega \setminus \Omega_0} K(x,y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_0$$

$$u \equiv 0 \quad \text{in} \quad \Omega_0$$

where $\Omega_0 := \{x \in \Omega |\, a_1(x) > 0\}$.

results

Existence of a refuge : i.e $\beta_{|\omega} \equiv 0$., C, 2012

For K, a_i, β as above then if $\omega \subset \Omega \setminus \Omega_0$ or $\omega \subset\subset \Omega_0$, there exists $\lambda^* < \lambda^* \in [-\infty; \infty]$, so that there exists a unique positive bounded stationary solution to (1) **if and only if** for all $\lambda^* < \lambda < \lambda^{**}$. there exists no bounded positive stationary solution u_λ of (1). Moreover, the map $\lambda \to u_\lambda$ is monotone increasing and we have

$$\forall x \in \bar{\Omega} \qquad \lim_{\lambda \to \lambda^{**},-} u_\lambda(x) = +\infty$$

$$\forall x \in \bar{\Omega} \qquad \lim_{\lambda \to \lambda^{*},+} u_\lambda(x) = u_\infty$$

where u_∞ is the unique non negative solution of

$$\int_{\Omega\setminus\Omega_0} K(x,y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_0$$

$$u \equiv 0 \quad \text{in} \quad \Omega_0$$

where $\Omega_0 := \{x \in \Omega | \, a_1(x) > 0\}$.

Survival in a heterogeneous habitat :

Known Characterisation for a refuge model :

$$\mathcal{K}_\Omega[u] + a(x)u - \beta(x)u^p = 0 \qquad (2)$$

When the operator $\mathcal{K}$ is replace by a diffusion operator $\mathcal{L} := a_{ij}(x)\partial_{ij} + b_i(x) + c(x)$ (Fraile, Garcia-Mellan, Lopez-Gomez, Cantrell-Cosner, Berestycki-Hamel- Roques,....) then there exists a unique positive solution to (2) **if and only if** $\lambda_1(\mathcal{L}_\Omega + a) < 0 < \lambda_1(\mathcal{L}_\omega + a)$.

Characterisation obtained (C, 2012)

There exists a unique positive solution to (2) **if and only if** $\mu_p(\mathcal{K}_\Omega + a) < 0 < \mu_p(\mathcal{K}_\omega + a)$, where μ_p is the generalized principal eigenvalue of the operator $\mathcal{K} + a$ defined by

$$\mu_p := \sup\{\mu \in \mathbb{R} \,|\, \exists \phi \in C(\Omega), \phi > 0, \text{ so that } \mathcal{K}[\phi] + (a+\mu)\phi \leq 0\}$$

.

Remarks :

- When $\beta > 0$ (2) its of KPP type and $\omega = \emptyset \implies \mu_p(\mathcal{K}_\omega + a) = +\infty$.
- **Main difficulties :** No compactness, Existence of eigen function is not guarantee.
- **Methods :**Sub and super-solution, Spectral Theory of non compact positive operator, ...

Proposition 1

(i) *Assume $\Omega_1 \subset \Omega_2$, then for the two operators*

$$\mathcal{L}_{\Omega_i}[u] + a(x)u := \int_{\Omega_i} K(x,y)u(y)\,dy - k(x)u + a(x)u$$

respectively defined on $C(\Omega_1)$ and $C(\Omega_2)$ we have

$$\mu_p(\mathcal{L}_{\Omega_1} + a(x)) \geq \mu_p(\mathcal{L}_{\Omega_2} + a(x)).$$

(ii) *Fix Ω and assume that $a_1(x) \geq a_2(x)$, then*

$$\mu_p(\mathcal{L}_{\Omega} + a_2(x)) \geq \mu_p(\mathcal{L}_{\Omega} + a_1(x)).$$

(iii) *$\mu_p(\mathcal{L}_{\Omega} + a(x))$ is Lipschitz continuous in $a(x)$. More precisely,*

$$|\mu_p(\mathcal{L}_{\Omega} + a(x)) - \mu_p(\mathcal{L}_{\Omega} + b(x))| \leq \|a(x) - b(x)\|_\infty$$

(iv) *Assume $\Omega_1 \subset \Omega_2$ and consider the two operators $\mathcal{L}_{\Omega_1}, \mathcal{L}_{\Omega_2}$. Assume that the corresponding principal eigenvalue are associated to a positive continuous principal eigenfunction. Then we have*

$$|\mu_p(\mathcal{L}_{\Omega_1} + a(x)) - \mu_p(\mathcal{L}_{\Omega_2} + a(x))| \leq C_0|\Omega_2 \setminus \Omega_1|,$$

where C_0 depends only on K.

Idea of the proofs

Existence

For $\mu_p < 0$, using an approximation criteria (C 2010), $\exists \phi > 0$, $\phi \in L^\infty$ so that for ϵ small :

$$\mathcal{K}_\Omega[\epsilon\phi_p] + a(x)\epsilon\phi_p - \beta\epsilon^p\phi_p^p \geq 0.$$

Due the presence of the refuge, the construction of the supersolution is based on

$$\psi_1 := \begin{cases} C_1\eta_1 & \text{in} \quad \Omega \setminus \omega_{\frac{\delta}{2}} \\ 0 & \text{elsewhere,} \end{cases} \qquad \psi_2 := \begin{cases} C_2\eta_2\Psi_\delta & \text{in} \quad \omega_\delta \\ 0 & \text{elsewhere.} \end{cases}$$

where $\omega_\delta := \{x \in \Omega \mid d(x;\omega) > \delta\}$, Ψ_δ is the positive continuous eigenfunction associated to t $\mu_p(\mathcal{K}_{\omega_\delta} + a_\epsilon)$, η_i characteristic function, C_1 and C_2 are positive constants.

- $\psi := sup(\psi_1, \psi_2)$ is a supersolution for C_1, C_2, δ well chosen.

Monotone Iteration scheme give the existence. Uniqueness bis obtained by sweeping method.

Non-existence

When $\mu_p(\mathcal{K}_\omega + a) \leq 0$ the argument reads as follows.

- On ω, u satisfies $\mathcal{L}_\Omega[u] + au = 0$, which implies that $\max_{\bar{\omega}} a < 0$ and we have

$$\mathcal{L}_\omega[u] + au = -\int_{\Omega\setminus\omega} K(x,y)u(y)\,dy \leq 0. \tag{3}$$

- Key observation $\mu_p(\mathcal{L}_\omega + a) \leq 0 < -\max_{\bar{\omega}} a \Longrightarrow$ eigenfunction ϕ associated to $\mu_p(\mathcal{L}_\omega + a)$.
- As a consequence, $\exists \phi^* >$ associated to $\mu_p(\mathcal{K}^*_\omega + a)$ the dual operator of $\mathcal{K}_\omega + a$.
- Multiply (3) by ϕ^* and integrating over ω it follows

$$\int_\omega \phi^*(x)\mathcal{K}_\omega[u](x)dx + au\phi^*(x)dx \leq -c_0 \int_\omega \phi^* \left(\int_{\Omega\setminus\omega} K(x,y)\,dy\right),$$

  which using Fubini leads to the contradiction

$$0 \leq -\mu_p(\mathcal{K}^*_\omega + a)\int_\omega \phi^* u \leq -c_0 \int_\omega \phi^* \left(\int_{\Omega\setminus\omega} K(x,y)\,dy\right) < 0.$$

Non-existence

When $\lambda_p \geq 0$, we argue as follow :

- From (2) there exists δ and c_0 so a solution u

$$\begin{cases} \inf_\Omega u \geq c_0, \\ \inf_{x\in\Omega}(k(x) - a(x) + \beta(x)u^{p-1}) \geq \delta. \end{cases}$$

- From monotone properties of $g(x,s) := (a - \beta(x)s^{p-1})$,
$\implies a - \beta u^{p-1} \leq a(x) - \beta c_0^{p-1} \leq a$.
- Set $\gamma(x) = a(x) - \beta(x)c_0^{p-1} \leq a(x)$. By construction,

$$\mu_p(\mathcal{K}_\Omega + \gamma(x)) \geq \mu_p(\mathcal{K}_\Omega + a(x)) \geq 0.$$

- From (2), we have $\mathcal{K}_\Omega[u] + \gamma u \geq \mathcal{K}_\Omega[u] + au - \beta u^p = 0$, with a strict inequality for any $x \in \Omega \setminus \omega$.
- Key point : There exists $\delta > 0$ and a positive continuous function ϕ so that $\inf_\Omega \phi > \delta$ and

$$\mathcal{K}_\Omega[\phi] + \gamma\phi \leq 0.$$

- From $\phi > \delta$ we can define

$$\tau^* := \inf\{\tau > 0 | u \leq \tau\phi\}.$$

and show that $\tau^* = 0$.

Idea of the proofs of the Theorem

Abscence of refuge

- First observation the existence of a positive solution relates to the sign of

$$\mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1).$$

- Check easily that for $\lambda \geq 0$ then $\mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1) \leq \mu_p(\mathcal{K}_\Omega + a_0) < 0$. Survival criteria $\Longrightarrow$ existence of a positive stationary solution for $\lambda \geq 0$.
- Key observation, Proposition 1 for all λ we have

$$\mu_p(\mathcal{K}_{\Omega\setminus\Omega_0} + a_0) = \mu_p(\mathcal{K}_{\Omega\setminus\Omega_0} + a_0 + \lambda a_1) \geq \mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1).$$

- Case $\mu_p(\mathcal{K}_{\Omega\setminus\Omega_0} + a_0) < 0 \Longrightarrow \mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1) < 0 \Longrightarrow$ existence of a solution for all λ and $\lambda^* = -\infty$
- Case $\mu_p(\mathcal{K}_{\Omega\setminus\Omega_0} + a_0) = 0$
  - Either $\mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1) < 0$ for all $\lambda \leq 0$ and $\lambda^* = -\infty$.
  - Or there exists $\exists\lambda_0 \leq 0$ so that $\mu_p(\mathcal{K}_\Omega + a_0 + \lambda_0 a_1) = 0$.

$$\lambda^* := \sup\{\lambda \mid \mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1) = 0\}.$$

- Case $\mu_p(\mathcal{K}_{\Omega\setminus\Omega_0} + a_0) > 0$.
  - For $\lambda << -1$ then $\mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1) > 0$. More precisely,

$$\liminf_{\lambda\to-\infty} \mu_p(\mathcal{K}_\Omega + a_0 + \lambda a_1) > 0.$$

Perspectives and future work

Conclusion

- We provide an simple heterogeneous dispersal process which takes into account both the starting point and the end point of the jumps
- We also provide a promising possible way to evaluate agricultural strategies on non targeted species which already have given some insight on the structure of the population.

future Work

- Better understanding of the heterogeneous dispersal process in unbounded domain
- Other distance/more general framework ?
- Remove the constraint on ω on the refuge dynamics
- Have a better understanding of λ^*, λ^{**} with respect to a_1. Optimisation
- Incorporate seasonality by introducing time dependant coefficient
- Unbounded death rates ?

Thank you for your attention